

Turbulent Flow and Heat Transfer

6.1 INTRODUCTION

The following are the characteristics of turbulent motion

- **Irregularity**

Complex variations of velocity and temperature with space and time (fluctuations) are the dominant characteristics of a turbulent flow. The irregular motion is generated due to random fluctuations (Fig. 6.1). It is postulated that the fluctuations inherently come from disturbances (such as, roughness of the solid surface) and they may be either reduced due to viscous damping or may grow by drawing energy from the free stream. At a Reynolds number less than the critical, the kinetic energy of flow is not enough to sustain the random fluctuations against the viscous damping and in such cases laminar flow continues to exist. At a Reynolds number somewhat higher than critical, the kinetic energy of flow supports the growth of fluctuations and transition to turbulence is induced.

- **Strong mixing**

The fluctuating turbulent motion promotes higher level of transfer of momentum, heat and mass; - is practically the most important feature. Turbulent flows appear to be random. Turbulent flows are not always free of coherent structures. The coherent structures component is periodic or at least repeatable.

- **Three-dimensional turbulent motion**

For a parallel flow, the axial velocity component is

$$u(y, t) = \bar{u}(y) + u'(\Gamma, t)$$

where y is the normal direction and Γ is any space variable.

Even if the bulk motion is parallel, the fluctuation u' being random varies in all directions. Now let us look at the continuity equation

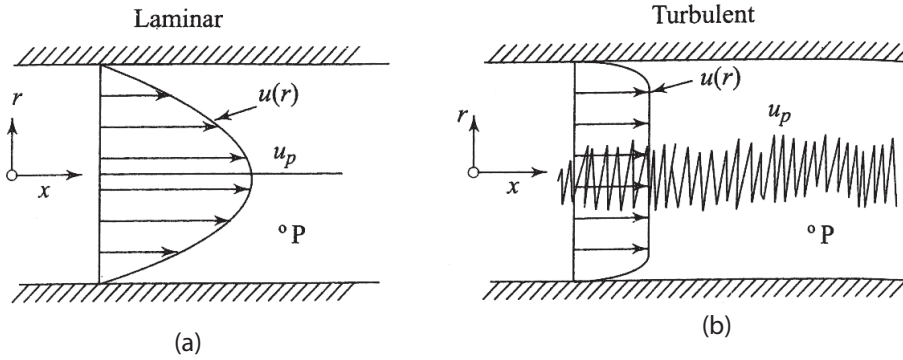


Figure 6.1 Presence of turbulence is appreciated from the velocity signal (u_p) at any point P in the flow field

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial u'}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Since $\frac{\partial u'}{\partial x} \neq 0$, the above equation depicts that y and z components of velocity exist even for the parallel flow if the flow is turbulent. We can write

$$\begin{aligned} u(y, t) &= \bar{u}(y) + u'(\Gamma, t) \\ v &= 0 + v'(\Gamma, t) \\ w &= 0 + w'(\Gamma, t) \end{aligned}$$

Turbulence is a three-dimensional process in which vortex stretching and tilting are essential mechanisms for energy exchange between different scales of motion.

- **Turbulent motion carries vorticity and consists of interacting eddies**
Wide spectrum of eddy sizes and corresponding fluctuation frequencies are shown in Fig. 6.2. In a turbulent flow, energy is fed from large scales mainly to smaller scales by nonlinear processes.
- The term *homogeneous turbulence* implies that the velocity fluctuations in the system are random.
- The average turbulent characteristics are independent of the position of the fluid, i.e., invariant to axis translation.
- If the velocity fluctuations are independent of the axis of reference (invariant to axis rotation and reflection), the restriction leads to *isotropic turbulence*, which by definition is always homogeneous.

Turbulence is generally damped by viscosity. One of the effects of viscosity on turbulence is to make the flow more homogeneous and less dependent on direction. **If the turbulence has the same structure quantitatively in**

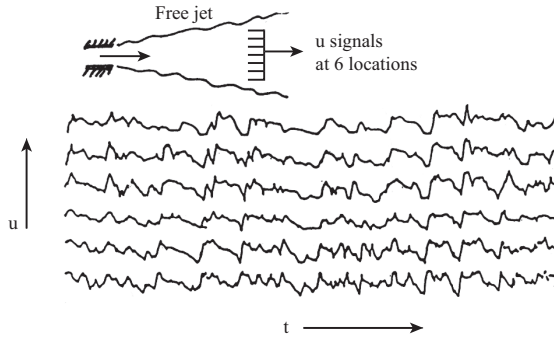


Figure 6.2 Wide spectrum of eddy sizes and corresponding frequencies

all parts of the flow field, the turbulence is said to be homogeneous. The turbulence is called isotropic if its statistical features have no directional preference and perfect disorder persists. Its velocity fluctuations are independent of the axis of reference, i.e., invariant to axis rotation and reflection. Isotropic turbulence is by its definition always homogeneous. In such a situation, the gradient of the mean velocity does not exist. The mean velocity is either zero or constant throughout. However, when the mean velocity has a gradient the turbulence is called anisotropic.

A little more discussion on homogeneous and isotropic turbulence is needed at this stage. The term *homogeneous turbulence* implies that the velocity fluctuations in the system are random. The average turbulent characteristic are independent of the position in the field, i.e., invariant to axis translation.

Consider the root mean square (rms) velocity fluctuations:

$$u' = \sqrt{\overline{u^2}}, \quad v' = \sqrt{\overline{v^2}}, \quad w' = \sqrt{\overline{w^2}}$$

In homogeneous turbulence, the rms values of u' , v' and w' can all be different, but each value must be uniform over the entire turbulent field. Even if the rms fluctuation of any component, say u 's are constant over the entire field, the instantaneous values of u may differ from point to point at any instant.

In addition to its homogeneous nature, if the velocity fluctuations are independent of the axis of reference, i.e., invariant to axis rotation and reflection, the situation leads to isotropic turbulence, which by definition as mentioned earlier, is always homogeneous.

In isotropic turbulence, fluctuations are independent of the direction of reference and

$$\sqrt{\overline{u^2}} = \sqrt{\overline{v^2}} = \sqrt{\overline{w^2}}$$

or

$$u' = v' = w'$$

Again, it is of relevance to say that even if the rms fluctuations at any point are same, their instantaneous values may differ from each other at any instant. Turbulent flow is also diffusive. In general, turbulence brings about better mixing of a fluid and produces an additional diffusive effect. The term *eddy diffusion* is often used to distinguish this effect from molecular diffusion. The effects caused by mixing are as if the viscosity is increased by a factor of 100 or more. At a large Reynolds number there exists a continuous transport of energy from the free stream to the large eddies. Then smaller eddies are formed continuously from the large eddies. Then from the large eddies smaller eddies are continuously formed. Near the wall, the smallest eddies dissipate energy and destroy themselves (Tennekes and Lumley [1]).

• Length scales of turbulence

The large scale motions with length scale L are unstable for large Reynolds numbers. The scale L is commensurate with the dimension of the flow domain. We shall try to understand smallest length scales of turbulent flows. At the smallest length scale, viscosity can be effective in smoothing out the velocity fluctuations. As mentioned, turbulence is generally damped by the viscosity ν and the main contribution to energy dissipation ϵ can be attributed to very small length scales of the order of the Kolmogorov-scale, l_s given by $(\nu^3/\epsilon)^{1/4}$. The parameters that govern the small scale motions include dissipation rate per unit mass, ϵ . The dimension of ϵ is $(m^2 s^{-3})$. The dimension of kinematic viscosity, ν is (m^2/s) . Making use of these parameters, it is possible to form length scale, time scale and velocity scale as

$$l_s = \left(\frac{\nu^3}{\epsilon} \right)^{1/4}, \quad t_s = (\nu/\epsilon)^{1/2} \quad \text{and} \quad v_s = (\nu\epsilon)^{1/4}$$

These are known as Kolmogorov scales of length, time and velocity. The Reynolds number formed by l_s and v_s ($l_s v_s/\nu$) = 1.

In the case of homogeneous turbulence, at high Reynolds numbers, the dissipation rate, ϵ equals the production-rate of turbulence and is of the order of u_k^3/L_k where u_k is similar to root-mean-square velocity of turbulence and L_k is the integral length scale of turbulence.

For the smaller length scales within the inertial range (where kinetic energy production rates due to external forcing and viscous dissipation rates are small) the turbulent motion becomes more and more homogeneous and isotropic even if the large scale motion is anisotropic. It is worth mentioning that ratio of largest to smallest length scales is L/L_s , which is proportional to $(Re)^{3/4}$.

6.2 CLASSICAL IDEALIZATION OF TURBULENT FLOWS

Since detailed descriptions of turbulent fluctuations are usually not of interest to engineers we take a statistical approach by averaging turbulence data. Different

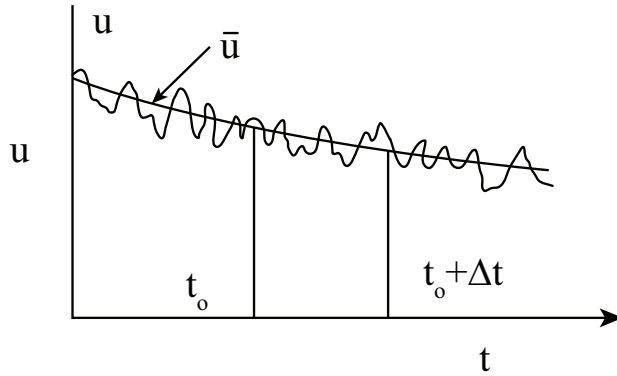


Figure 6.3 Mean motion and fluctuations

trends of variation of the the mean and fluctuating components are shown in Figs. 6.3 and 6.4. Fig 6.4(b) reveals an unsteady mean motion. The mean velocity has a time period t_2 and a large time period signifies low frequency oscillation of the mean motion. The high frequency oscillations have a time period of t_1 .

Statistical quantities may be calculated as

$$u_i = \bar{u}_i + u'_i, p = \bar{p} + p', \bar{u} = \frac{1}{\Delta t} \int_{t_o}^{t_o + \Delta t} u dt \quad (6.1)$$

However the fluctuating components do not bring about the bulk displacement of a fluid element. The instantaneous displacement is $u' dt$ and if that is indeed not responsible for the bulk motion, we can conclude that

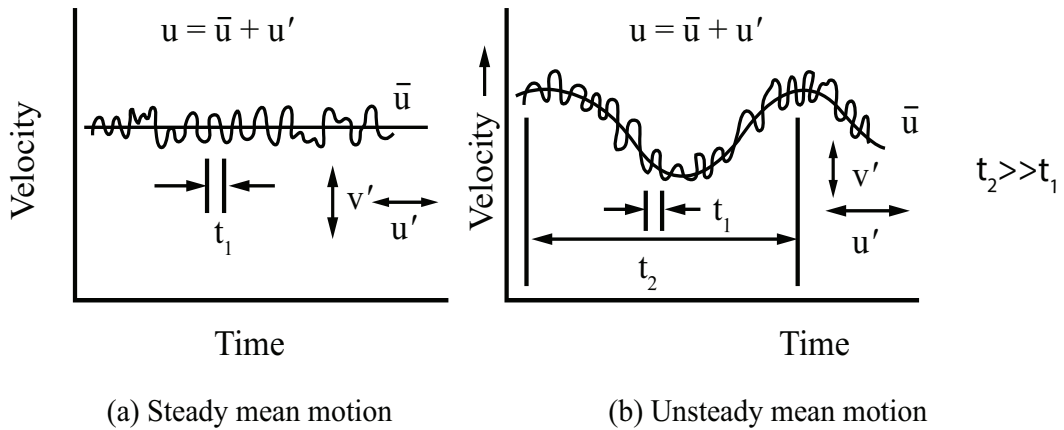


Figure 6.4 Steady and unsteady mean motions in a turbulent flow

$$\int_{t_o}^{t_o + \Delta t} u' dt = 0 \quad (6.2)$$

Due to the interaction of fluctuating components, macroscopic momentum transport takes place. Therefore, interaction effect between two fluctuating components over long period is nonzero and this yields

$$\int_{t_0}^{t_0+\Delta t} u' v' dt \neq 0 \quad (6.3)$$

We take time average of these two integrals and write

$$\overline{u'} = \frac{1}{\Delta t} \int_{t_0}^{t_0+\Delta t} u' dt = 0 \quad (6.4)$$

and

$$\overline{u'v'} = \frac{1}{\Delta t} \int_{t_0}^{t_0+\Delta t} u' v' dt \neq 0 \quad (6.5)$$

Now, we can make a general statement with any two fluctuating parameters, say, with f' and g' as (f' and g' can be vectors or passive scalars)

$$\overline{f'} = \overline{g'} = 0 \quad \frac{\partial \overline{f'}}{\partial s} = \frac{\partial^2 \overline{f'}}{\partial s^2} = 0 \quad (6.6)$$

and

$$\overline{f'g'} \neq 0 \quad \text{and} \quad \frac{\partial \overline{(f'g')}}{\partial s} \neq 0 \quad (6.7)$$

We shall state some rules of operation on mean time-averages herein. If f and g are two dependent variables and if s denotes any one of the independent variables x, y, z, t then

$$\frac{\partial \overline{f}}{\partial s} = \frac{\partial \overline{f}}{\partial s}; \quad \overline{\int f ds} = \int \overline{f} ds \quad (6.8)$$

The Reynolds decomposition may be expressed as

$$u = \overline{u} + u', \quad v = \overline{v} + v', \quad w = \overline{w} + w', \quad p = \overline{p} + p' \quad (6.9)$$

Plugging in continuity, we get

$$\nabla \cdot \overline{u_i} = 0 \quad (6.10)$$

and

$$\nabla \cdot u'_i = 0 \quad (6.11)$$

Introduction of Reynolds decomposition into the Navier-Stokes equations and subsequent averaging and application of the laws of statistics leads to the appearance of

turbulence correlations (turbulent or Reynolds stresses). For example, if we perform the aforesaid exercise on the x momentum equation, we obtain

$$\rho \left\{ \frac{\partial \bar{u}}{\partial t} + \frac{\partial (\bar{u} \bar{u})}{\partial x} + \frac{\partial (\bar{u} \bar{v})}{\partial y} + \frac{\partial (\bar{u} \bar{w})}{\partial z} \right\} = -\frac{\partial \bar{p}}{\partial x} + \mu \nabla^2 \bar{u} - \rho \left[\frac{\partial \overline{u'^2}}{\partial x} + \frac{\partial \overline{u'v'}}{\partial y} + \frac{\partial \overline{u'w'}}{\partial z} \right] \quad (6.12)$$

Introducing simplifications arising out of continuity equation we shall obtain

$$\rho \frac{\partial \bar{u}}{\partial t} + \rho \left[\bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y} + \bar{w} \frac{\partial \bar{u}}{\partial z} \right] = -\frac{\partial \bar{p}}{\partial x} + \mu \nabla^2 \bar{u} - \rho \left[\frac{\partial \overline{u'^2}}{\partial x} + \frac{\partial \overline{u'v'}}{\partial y} + \frac{\partial \overline{u'w'}}{\partial z} \right] \quad (6.13)$$

Performing a similar treatment on y and z momentum equations, we obtain the y and z momentum equations in the form

$$\rho \frac{\partial \bar{v}}{\partial t} + \rho \left[\bar{u} \frac{\partial \bar{v}}{\partial x} + \bar{v} \frac{\partial \bar{v}}{\partial y} + \bar{w} \frac{\partial \bar{v}}{\partial z} \right] = -\frac{\partial \bar{p}}{\partial y} + \mu \nabla^2 \bar{v} - \rho \left[\frac{\partial \overline{u'v'}}{\partial x} + \frac{\partial \overline{v'^2}}{\partial y} + \frac{\partial \overline{v'w'}}{\partial z} \right] \quad (6.14)$$

$$\rho \frac{\partial \bar{w}}{\partial t} + \rho \left[\bar{u} \frac{\partial \bar{w}}{\partial x} + \bar{v} \frac{\partial \bar{w}}{\partial y} + \bar{w} \frac{\partial \bar{w}}{\partial z} \right] = -\frac{\partial \bar{p}}{\partial z} + \mu \nabla^2 \bar{w} - \rho \left[\frac{\partial \overline{u'w'}}{\partial x} + \frac{\partial \overline{v'w'}}{\partial y} + \frac{\partial \overline{w'^2}}{\partial z} \right] \quad (6.15)$$

It is to be noted that the terms containing prime symbols were not in the original NS equations.

$$\sigma_T = \begin{bmatrix} \sigma'_{xx} & \tau'_{xy} & \tau'_{xz} \\ \tau'_{xy} & \sigma'_{yy} & \tau'_{yz} \\ \tau'_{xz} & \tau'_{yz} & \sigma'_{zz} \end{bmatrix} = -\rho \begin{bmatrix} \overline{u'^2} & \overline{u'v'} & \overline{u'w'} \\ \overline{u'v'} & \overline{v'^2} & \overline{v'w'} \\ \overline{u'w'} & \overline{v'w'} & \overline{w'^2} \end{bmatrix} \quad (6.16)$$

σ_T is the Reynolds stress tensor and written in compact form as $-\rho \overline{u'_i u'_j}$

$$\sigma_{xx} = -p + 2\mu \frac{\partial \bar{u}}{\partial x} - \rho \overline{u'^2} \quad (6.17)$$

$$\tau_{xy} = \mu \left(\frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x} \right) - \overline{\rho u' v'} \quad (6.18)$$

The averaged equations may be written, in a compact form, as

$$\frac{\partial \bar{u}_i}{\partial t} + u_j \frac{\partial \bar{u}_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_j} \left(\nu \frac{\partial \bar{u}_i}{\partial x_j} - \overline{u'_i u'_j} \right) \quad (6.19)$$

$$\frac{\partial \bar{u}_i}{\partial x_i} = 0 \quad (6.20)$$

We have more unknowns than number of available equations. The modified system of equations cannot be closed within itself unless empirical relations are supplied from experiments to correlate the fluctuating components with the mean motion. This is termed as the closure problem.

In solving the closure problem, the turbulent stresses should be determined by using a turbulence model.

This section discusses only models for turbulent motions. The subgrid-scale models for large-eddy simulations are not included in this discussion.

$$-\overline{u'_i u'_j} = \nu_t \left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right) - \frac{2}{3} k \delta_{ij} \quad (6.21)$$

The term ν_t is turbulent (eddy) viscosity. The term involving the Kronecker delta δ_{ij} in Eq. (6.21) is perhaps a somewhat unfamiliar addition to the eddy-viscosity expression. It is necessary to make the expression applicable also to normal stresses (when $i = j$). The first part of Eq. (6.21) involving the velocity gradients would yield the normal stresses.

$$\overline{u'^2} = -2\nu_t \frac{\partial \bar{u}}{\partial x}, \quad \overline{v'^2} = -2\nu_t \frac{\partial \bar{v}}{\partial y}, \quad \overline{w'^2} = -2\nu_t \frac{\partial \bar{w}}{\partial z} \quad (6.22)$$

The sum of the stresses is zero because of the continuity equation. However, all normal stresses are by definition positive quantities, and their sum is twice the kinetic energy k of the fluctuating motion:

$$k = \frac{1}{2} \left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2} \right) \quad (6.23)$$

Inclusion of the second part of the eddy viscosity expression (Eq. (6.21)) assures that the sum of the normal stresses is equal to $2k$. The normal stresses act like pressure forces (i.e., perpendicular to the faces of a control volume). The second part of Eq. (6.21) constitutes a pressure because, like pressure, energy k is a scalar quantity. Therefore, when Eq. (6.21) is used to eliminate $\overline{u'_i u'_j}$ in the momentum equation, this second part can be absorbed by the pressure-gradient term so that in effect the static pressure is replaced as unknown quantity by the pressure $\bar{p} + \frac{2}{3}k$. Therefore the appearance of k in Eq. (6.21) does not necessitate the determination

of k , it is the distribution of the eddy viscosity ν_t only that has to be determined. In this context, k can be linked to the intensity of turbulence which is given by

$$I = \sqrt{\frac{1}{3}(\overline{u'^2} + \overline{v'^2} + \overline{w'^2})} / U_\infty \quad (6.24)$$

We can also write non-dimensional kinetic energy as

$$\frac{k}{U_\infty^2} = 1.5 I^2$$

6.3 REYNOLDS AVERAGED FORM OF ENERGY EQUATION

Let us consider a three-dimensional situation,

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} = \alpha \left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right] \quad (6.25)$$

Using Reynolds decomposition of velocity and temperature

$$u = \bar{u} + u', \quad v = \bar{v} + v', \quad w = \bar{w} + w' \quad T = \bar{T} + T' \quad (6.26)$$

we get

$$\begin{aligned} \bar{u} \frac{\partial \bar{T}}{\partial x} + \bar{v} \frac{\partial \bar{T}}{\partial y} + \bar{w} \frac{\partial \bar{T}}{\partial z} &= \frac{\partial}{\partial x} \left(\alpha \frac{\partial \bar{T}}{\partial x} - \overline{u'T'} \right) \\ &+ \frac{\partial}{\partial y} \left(\alpha \frac{\partial \bar{T}}{\partial y} - \overline{v'T'} \right) + \frac{\partial}{\partial z} \left(\alpha \frac{\partial \bar{T}}{\partial z} - \overline{w'T'} \right) \end{aligned} \quad (6.27)$$

The terms $\overline{u'T'}$, $\overline{v'T'}$, and $\overline{w'T'}$ thus, cause additional heat flux in the x , y and z directions respectively, due to turbulent motion. The total heat flux in the three directions will therefore be given by

$$\begin{aligned} q_x'' &= -\rho c_p \left(\alpha \frac{\partial \bar{T}}{\partial x} - \overline{u'T'} \right); \quad q_y'' = -\rho c_p \left(\alpha \frac{\partial \bar{T}}{\partial y} - \overline{v'T'} \right); \\ q_z'' &= -\rho c_p \left(\alpha \frac{\partial \bar{T}}{\partial z} - \overline{w'T'} \right) \end{aligned} \quad (6.28)$$

As in the case of turbulent transport of momentum, it is convenient to define an eddy viscosity or turbulent viscosity, to study the turbulent transport of thermal energy, a turbulent thermal diffusivity α_t can be defined.

The total heat flux in x, y and z directions can therefore be given as

$$\frac{q_x''}{\rho c_p} = -(\alpha + \alpha_t) \frac{\partial \bar{T}}{\partial x}, \quad \frac{q_y''}{\rho c_p} = -(\alpha + \alpha_t) \frac{\partial \bar{T}}{\partial y}, \quad \frac{q_z''}{\rho c_p} = -(\alpha + \alpha_t) \frac{\partial \bar{T}}{\partial z} \quad (6.29)$$

Like eddy viscosity, α_t is not a fluid property but depends on the state of turbulence. In fact the Reynolds analogy between heat and momentum transport suggests

$$\alpha_t = \frac{\nu_t}{\sigma_t} \quad (6.30)$$

The denominator σ_t is called turbulent Prandtl number. Experiments have shown that σ_t varies very little across the flow. Many models make use of σ_t as a constant. For the flow of air a value of 0.9 may be chosen.

Finally the transport equation (6.27) may be written as

$$\begin{aligned} \bar{u} \frac{\partial \bar{T}}{\partial x} + \bar{v} \frac{\partial \bar{T}}{\partial y} + \bar{w} \frac{\partial \bar{T}}{\partial z} &= \frac{\partial}{\partial x} \left[(\alpha + \alpha_t) \frac{\partial \bar{T}}{\partial x} \right] \\ &+ \frac{\partial}{\partial y} \left[(\alpha + \alpha_t) \frac{\partial \bar{T}}{\partial y} \right] + \frac{\partial}{\partial z} \left[(\alpha + \alpha_t) \frac{\partial \bar{T}}{\partial z} \right] \end{aligned} \quad (6.31)$$

6.4 PRANDTL'S MIXING LENGTH

In the case of turbulent flow, velocity, pressure and temperature at a fixed point in space do not remain constant with time but undergo very irregular fluctuations of high frequency. The velocity of the fluid in turbulent flow can be represented by two components

- (i) mean motion
- (ii) fluctuating component (or eddying motion)

The velocity of fluid is given by

$$u = \bar{u} + u', \quad v = \bar{v} + v', \quad w = \bar{w} + w'$$

Similarly,

$$p = \bar{p} + p', \quad T = \bar{T} + T'$$

However, we must always consider that the fluctuating components do not bring about the bulk displacement of a fluid element. Consider continuity equation in two dimensions: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$. If we substitute $u = \bar{u} + u'$ and $v = \bar{v} + v'$ and then perform time averaging, we shall obtain

$$\frac{\partial(\overline{\bar{u} + u'})}{\partial x} + \frac{\partial(\overline{\bar{v} + v'})}{\partial y} = 0$$

or

$$\left[\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} \right] + \left[\frac{\partial \overline{u'}}{\partial x} + \frac{\partial \overline{v'}}{\partial y} \right] = 0 \quad (6.32)$$